

# On the relation between the spin and the magnetic moment of the proton

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## Abstract

In the context of the quark model of hadrons the spin and the magnetic moment of proton can not be taken proportional. This is in contradiction with the widely used relation between these two properties of the proton. This apparent difficulty is addressed by the most elementary notions of the relevant physics. In particular it is emphasized that the widely used relation is only valid in the lowest orders of perturbation, in which transitions between different baryons do not occur. For other processes where such transitions do occur, such as inelastic scattering off the protons, the quark model relation for the magnetic moment is used to give an estimation for the amplitude of transition between states with different total spins.

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It is known that for composite systems, whenever the charge-to-mass ratio ( $q_a/m_a$ ) are not the same for the constituents, the magnetic moment of the system is not proportional to its total angular momentum (see e.g. p. 187 of [1]). Further, in atomic physics it is known that, due to different gyromagnetic factors (the so-called  $g$ -factors) of the orbital angular momentum and spin, the magnetic moment of atoms and their total angular momentum are not proportional (see e.g. p. 301 of [2]). It is the purpose of this note to address a less emphasized similar phenomena for the composite states of the hadron physics in the context of the quark model [3, 4].

According to the quark model the hadrons are made up of quarks. Each quark is specified by its mass, charge, spin, flavor, and color. The states of baryons with the lowest masses are symmetric combinations in the space of “flavor  $\otimes$  spin”. For example, the spin-up proton state is presented by [3]

$$\begin{aligned} |p\uparrow\rangle = \frac{1}{3\sqrt{2}} & \left( |udu - duu\rangle |\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow\rangle \right. \\ & + |uud - udu\rangle |\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow\rangle \\ & \left. + |uud - duu\rangle |\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\rangle \right), \end{aligned} \quad (1)$$

in which  $|u\rangle$  and  $|d\rangle$  represent the up and down flavors of quarks, respectively. A similar expression can be given for the spin-down proton state,  $|p\downarrow\rangle$ . It is easy to check that both states  $|p\uparrow\rangle$  and  $|p\downarrow\rangle$  are simultaneous eigenstates of the square and the  $z$ -component of the total-spin operator

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3, \quad (2)$$

with eigenvalues  $\frac{3}{4}\hbar^2$  and  $\pm\frac{1}{2}\hbar$ , respectively.

In the context of the quark model, the magnetic moment of a baryon state is defined as the vector sum of the magnetic moments of its three constituent quarks [3, 4]:

$$\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3, \quad (3)$$

in which (SI units)

$$\boldsymbol{\mu}_a = \frac{Q_a}{m_a} \mathbf{S}_a, \quad a = 1, 2, 3 \quad (4)$$

with  $Q_a$ ,  $m_a$ , and  $\mathbf{S}_a$  representing the electric charge, the constituent mass, and the spin operator of the  $a$ 'th quark, respectively. As is evident, the two vectors (2) and (3), even after equating the constituent masses  $m_a$ 's, are not parallel. Further, one can easily check that neither  $|p\uparrow\rangle$  nor  $|p\downarrow\rangle$  are eigenstates of the  $z$ -component of the vector (3). The reason simply is that the electric charges of up and down flavors are different. The commutation relations

$$[\mu_i, S_i] = 0, \quad (5)$$

$$[\mu_i, \mathbf{S} \cdot \mathbf{S}] \neq 0, \quad i = x, y, z \quad (6)$$

mean that, for example, the  $\mu_z$  and  $S_z$  can be determined simultaneously, but not in the basis in which the total-spin is given. In fact, in the context of the

quark model it is known that the matrix element of the magnetic moment operator between different hadronic states, the so-called transition magnetic moment, can give an estimation for the amplitude of the transition between states with different total spins as well. For example, the matrix element  $\langle p | \mu_z | \Delta^+ \rangle \neq 0$  can be used to estimate the amplitude at the corresponding resonance [5] (see [6, 7] for a presentation in the nowadays notation).

The simple observations above should be contrasted with the widely used expression, that the relation between the spin and magnetic moment of proton is given by [3, 4, 8, 9, 10]

$$\boldsymbol{\mu}_p = g_p \frac{e}{2M_p} \mathbf{S}_p, \quad g_p = 5.59 \quad (7)$$

saying that

- 1) the spin and magnetic moment of the proton are parallel,
- 2) the total spin squared ( $\mathbf{S} \cdot \mathbf{S}$ ), as well as the  $z$ -components of the total spin and the total magnetic moment ( $S_z$  and  $\mu_z$ , respectively) can be determined simultaneously.

So the above mentioned transitions between states with different spins would have been forbidden by relations like (7). It is the purpose of this work to clarify these within the context of the quark model, and to explore the domain of validity of (7). In particular, it is emphasized that the relation (7) should be interpreted as a relation between the matrix elements of the spin and the magnetic moment in the lowest order of perturbation, where transition between different baryon states does not occur. Hence if the energy due to the coupling of the magnetic moment with the external field is so high that transition from one baryon state to another one is possible, the relation (7) is not valid anymore.

Here only baryons with the quark content “uud” (spin-1/2 p and spin-3/2  $\Delta^+$  [3, 4]) are studied. Though it should be emphasized that a more or less similar reasoning shows that the present analysis applies to other hadronic states as well.

As an example, the  $\Delta^+$  state with  $s_z = \frac{3}{2}\hbar$  is given by [3]

$$\left| \Delta^+, +\frac{3}{2} \right\rangle = \frac{1}{\sqrt{3}} |uud + udu + duu\rangle |\uparrow\uparrow\uparrow\rangle \quad (8)$$

One can give similar expressions for  $\Delta^+$  states with  $s_z = \frac{1}{2}\hbar, -\frac{1}{2}\hbar, -\frac{3}{2}\hbar$ . If all other interactions had been turned off, the two p states and the four  $\Delta^+$  states would have been the eigenstates of the mass operator

$$M_0 = m_1 + m_2 + m_3 \quad (9)$$

with the eigenvalue  $2m_u + m_d$ . So the proton and  $\Delta^+$  would have equal masses. In the context of the quark model, a strong version of the hyperfine effect is

responsible for the mass splitting between these two baryons [3]. In the case of our interest, assuming  $m_u = m_d = m$ , we have

$$M'_0 = 3m + \frac{A}{m^2} (\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_1 \cdot \mathbf{S}_3) \quad (10)$$

in which the free constant  $A$  is tuned such that the best fit to the mass data is achieved [3]. Since the above hyperfine term commutes with the square of total spin,  $\mathbf{S} \cdot \mathbf{S}$ , the mass degeneracy between the six states of the proton and  $\Delta^+$  is reduced to a degeneracy between the two states of the proton and a degeneracy between the four states of  $\Delta^+$ :

$$\begin{aligned} M_p &= 938 \frac{\text{MeV}}{c^2}, \\ M_{\Delta^+} &= 1232 \frac{\text{MeV}}{c^2}. \end{aligned} \quad (11)$$

The unperturbed rest mass operator in the sub-space of the “uud” quark content and in the basis of the simultaneous eigenvectors of  $\mathbf{S} \cdot \mathbf{S}$  and  $S_z$  is in the form

$$M'_0 = \begin{pmatrix} M_p & 0 & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{\Delta^+} & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{\Delta^+} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{\Delta^+} & 0 \\ 0 & 0 & 0 & 0 & 0 & M_{\Delta^+} \end{pmatrix}. \quad (12)$$

Let us now address the issue of the magnetic moment. The magnetic moment of particles are usually measured by monitoring the behavior of them in an external magnetic field  $\mathbf{B}(\mathbf{r}, t)$ . Such a field results in an interaction term

$$\delta H = -\boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{r}, t). \quad (13)$$

Common examples of such experiments are the Stern-Gerlach or the spin resonance set-ups. As mentioned before, baryonic states of definite total spin squared for which the constituent flavors have different electric charges, are not eigenstates of a component of the total magnetic moment. This means that the perturbation Hamiltonian (13) has off diagonal elements. Subsequently, one should expect that the transitions between different “uud” states are possible. In other words, for the matrix elements which are responsible for the previously mentioned proton- $\Delta$  transition [5, 6, 7], one can directly show that

$$\langle p, s_z | \boldsymbol{\mu} \cdot \mathbf{B} | \Delta^+, s_z \rangle \neq 0, \quad \mathbf{B} \parallel \hat{\mathbf{z}}, \quad (14)$$

$$\langle p, s_z | \boldsymbol{\mu} \cdot \mathbf{B} | \Delta^+, s'_z \rangle \neq 0, \quad \mathbf{B} \perp \hat{\mathbf{z}} \text{ \& } s_z \neq s'_z, \quad (15)$$

in which  $\boldsymbol{\mu}$  is given by (3). One should notice that by using  $\boldsymbol{\mu}_p$  of (7) instead of (3) in the above, the matrix elements which are responsible for the transition  $p \rightleftharpoons \Delta^+$  would vanish, simply due to the fact that proton belongs to the spin-1/2 subspace while  $\Delta^+$  belongs to the spin-3/2 one. In the perturbative regime,

$|\mu B| \ll (M_{\Delta^+} - M_p) c^2$ , however, up to first order in perturbation only those matrix elements of  $\delta H$  contribute which correspond to states inside each of the degenerate blocks of the original Hamiltonian [8, 9]. For the proton block, it can be directly shown that

$$\langle p, s_z | \boldsymbol{\mu} | p, s'_z \rangle = \frac{2}{\hbar} \mu_p \langle p, s_z | \mathbf{S} | p, s'_z \rangle, \quad (16)$$

where

$$\mu_p := \langle p \uparrow | \mu_z | p \uparrow \rangle. \quad (17)$$

Equation (17) is precisely the same that is given as the strength of the magnetic moment of proton in the context of the quark model of hadrons [3]. It is seen that, as far as the lowest order of perturbation is concerned, one can employ (7), in which the spin and the magnetic moment of proton are parallel.

Let us elaborate it a little more. In general the magnetic moment has two origins, the spin and the orbital angular momentum. Regarding the proton, the contribution from the latter does not exist, as the quarks inside the proton are in a state of zero orbital angular momentum. Hence (3) and (4) follow. It could then be expected that the magnetic moment of the proton should be a c-number times the spin of the proton, as the proton is spherically symmetric from the point of view of the orbital motion of its constituent quarks. This would be true if the magnetic moment and the spin were vectors with c-number components, and in fact it is true for the expectation value of the magnetic moment and the spin, as stated in (16). Equation (16) is even more than that. Not only the expectation values, but the general matrix elements of the magnetic moment and the spin are proportional by a single proportionality constant, provided one is restricted to the subspace corresponding the proton. The point is that the magnetic moment operator is not equal to a c-number times the spin operator. The magnetic moment operator is a c-number times the spin operator, in the subspace of proton, and another c-number times the spin operator in the subspace of  $\Delta^+$ . Moreover, the magnetic moment operator does have nonzero matrix elements between these two subspaces, while the spin operator is not so:

$$\begin{aligned} \langle p, s_z | \boldsymbol{\mu} | p, s'_z \rangle &= c_p \langle p, s_z | \mathbf{S} | p, s'_z \rangle, \\ \langle \Delta^+, s_z | \boldsymbol{\mu} | \Delta^+, s'_z \rangle &= c_{\Delta^+} \langle \Delta^+, s_z | \mathbf{S} | \Delta^+, s'_z \rangle, \\ \langle \Delta^+, s_z | \boldsymbol{\mu} | p, s'_z \rangle &\neq 0, \quad \text{in general.} \end{aligned} \quad (18)$$

This is a manifestation of the Wigner-Eckart theorem.

As mentioned before, turning on the interaction in principle mixes the two blocks corresponding to the proton and  $\Delta^+$ . In order that this mixing be significant, the matrix elements of the perturbation Hamiltonian should be comparable with the difference between the eigenvalues of the unperturbed Hamiltonian. Let us have an estimate for the threshold value of the magnetic field, corresponding to such a mixing

$$(\mu_p B) \sim (M_{\Delta^+} - M_p) c^2, \quad (19)$$

resulting in

$$B \sim 3 \times 10^{15} \text{ T}. \quad (20)$$

Even at the surface of a magnetar, the magnetic field reaches only up to  $10^{11}$  T. So, one can use the relation (7) with extreme safety in almost all practical cases. But why almost?

As mentioned before, in the situations in which transitions between the baryon states are possible, the lowest orders of perturbation, by which the relation (7) is justified, are insufficient. In fact, in the processes of scattering off the protons we are faced with this situation. Let us make an estimation on the possibility of generation of the threshold value mentioned above in the scattering of electrons off the rest protons. The magnetic field of an ultra-relativistic electron at transverse distance  $b$  is given by [1]

$$B_e \sim \frac{\mu_0 c}{4\pi} \gamma \frac{e}{b^2}, \quad (21)$$

where  $\gamma$  is the Lorentz factor. The corresponding de Broglie wavelength is

$$\begin{aligned} \lambda &= \frac{h}{\gamma m c}, \\ &\sim \frac{2 \times 10^{-12}}{\gamma} \text{ m}, \end{aligned} \quad (22)$$

showing that for an electron of the Lorentz factor 2000 (energy equal to 1 GeV), this wavelength is of the order  $10^{-15}$  m. So such an electron can be close to a proton down to such a distance. It is then seen that a threshold field can be produced by such an electron at a distance of the order  $10^{-15}$  m. In fact, in inelastic scattering experiments, transitions of the type  $e p \rightarrow e \Delta \rightarrow e p \pi^0$  are considered as the ones with the modest amount of the energy transfer between the incoming electron and the proton [4].

If the magnetic field is so large that (19) holds, then the mass matrix would be no longer like (12). It would have non-diagonal terms (in the basis consisting of  $p$  and  $\Delta^+$  states). So  $p$  and  $\Delta^+$  would no longer be eigenstates of mass. The mass eigenstates would be mixtures of  $p$  and  $\Delta^+$ . One would then expect an oscillation between  $p$  and  $\Delta^+$ . So a very high magnetic field (coming from whatever source, an inelastic scattering for example) could induce an oscillation between the  $p$  and  $\Delta^+$  states. The off-diagonal matrix elements which are responsible for such an oscillation are easily calculated. For example, consider a magnetic field in the  $z$  direction. One has,

$$\left\langle \Delta^+, s_z = \frac{\hbar}{2} \right| (-\mu_z B) \left| p, s_z = \frac{\hbar}{2} \right\rangle = -\frac{\sqrt{2} B \hbar}{3} \left( \frac{Q_u}{m_u} - \frac{Q_d}{m_d} \right), \quad (23)$$

which vanishes if and only if the charge to mass ratio is the same for the up and down quarks. So the oscillation occurs, if and only if the charge to mass ratio is different for the up and down quarks, or equivalently if and only if the magnetic moment is not parallel to the spin.

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